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Axial heat conduction effects in natural convection along a vertical cylinder

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INTRODUCTION

MANY TRANSPORT processes occur in nature and in industrial applications in which the transfer of heat is governed by the process of natural convection. Natural convection arises in fluids when the temperature changes cause density variations leading to buoyancy forces. An excellent review of natural convection flows has been given by Ede [1]. Recently, Minkowycz and Sparrow [2, 3], Cebeci [4], and Aziz and Na [5] have studied the steady, laminar, incompressible, natural convection flow over a vertical cylinder using a local nonsimilarity method, a finite-difference scheme, and an improved perturbation method, respectively. However, they did not take into account the effect of axial heat conduction for small Prandtl number. It is known that the axial heat conduction effect becomes important for low-Prandtl-number fluids such as a liquid metal.

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The aim of the present analysis is to study the effect of axial heat conduction on the steady, laminar, incompressible, natural convection flow over a vertical cylinder. The partial differential equations governing the flow have been solved numerically using an implicit finite-difference scheme in combination with the quasilinearization technique [6]. The results have been compared with the available results [2–5].

GOVERNING EQUATIONS

We consider a thin, vertical cylinder of radius r_0 which is situated in a quiescent environment having temperature T_∞. The surface of the cylinder is maintained at a uniform temperature T_w. The axial and radial coordinates are taken to be x and r, with x measuring the distance along the centerline of the cylinder from its bottom end and r measuring normal to the axis of the cylinder. The gravitational force acts in the opposite direction to x. The fluid is assumed to have constant

NOMENCLATURE

F, F_w	dimensionless streamfunction and mass transfer parameter, respectively
F''_w, G'_w	skin friction and heat transfer parameters, respectively
g, G	gravitational acceleration and dimensionless temperature, respectively
Gr, Gr_x	Grashof number and local Grashof number, respectively
Nu, Pr	Nusselt number and Prandtl number, respectively
r, x	radial and axial coordinates, respectively
r_0	radius of cylinder
T	temperature
u, v	velocity components in x- and r-directions, respectively.

Greek symbols

α, β	thermal diffusivity of the fluid and volumetric coefficient of thermal expansion, respectively
ξ, η	transformed coordinates
λ, ν, ψ	axial heat conduction parameter, kinematic viscosity, and dimensional streamfunction, respectively.

Superscript

$'$	differentiation with respect to η.

Subscripts

x, r, ξ	derivatives with respect to x, r and ξ, respectively
w, ∞	conditions at the wall and in the free stream, respectively.

properties with a linear density–temperature relationship for the buoyancy term. The boundary-layer equations taking into account the effect of axial heat conduction represented by αT_{xx} in the energy equation can be expressed as [2–5]

$$(ru)_x+(rv)_r = 0 \tag{1a}$$

$$uu_x+vu_r = g\beta(T-T_\infty)+r^{-1}v(ru_r)_r \tag{1b}$$

$$uT_x+vT_r = \alpha[T_{xx}+r^{-1}(rT_r)_r]. \tag{1c}$$

The boundary conditions are

$$u=0,\quad v=v_w,\quad T=T_w \quad \text{at} \quad r=r_0 \tag{2a}$$

$$u\to 0,\quad T\to T_\infty \quad \text{as} \quad r\to\infty. \tag{2b}$$

$$T_x\to 0 \quad \text{as} \quad x\to\infty. \tag{2c}$$

We apply the following transformations

$$\xi = 2(x/r_0)^{1/4}(Gr)^{-1/4},\quad \eta = (Gr)^{1/4}(r^2r_0^{-2}-1)/2(x/r_0)^{1/4}$$

$$\psi = 4vr_0(x/r_0)^{3/4}(Gr)^{1/4}F(\xi,\eta),\quad ru=\psi_r,\quad rv=-\psi_x$$

$$(T-T_\infty)/(T_w-T_\infty) = G(\xi,\eta),\quad Gr = g\beta(T_w-T_\infty)r_0^3/4v^2$$

to the set of equations (1) and we find that equation (1a) is satisfied identically and equations (1b) and (1c) reduce to

$$[(1+\xi\eta)F'']'+3FF''-2F'^2+G = \xi(FF'_\xi-F''F_\xi) \tag{3a}$$

$$Pr^{-1}[(1+\xi\eta)G']'+3FG'+\lambda(\eta^2G''-2\xi\eta G'_\xi+5\eta G' -3\xi G_\xi+\xi^2G_{\xi\xi}) = \xi(F'G_\xi-G'F_\xi). \tag{3b}$$

After transformation, the boundary conditions become

$$\text{at}\quad \eta=0:\quad F=F_w,\quad F'=0,\quad G=1$$

$$\text{at}\quad \eta=\infty:\quad F'=G=0;\quad \text{at}\quad \xi=\infty:\quad G_\xi=0 \tag{4}$$

where

$$F_w = -(v_wx^{-1/4}/v)/[g\beta(T_w-T_\infty)/4v^2]^{1/4}$$

$$\lambda = (1/8)(Gr_xPr^2)^{-1/2} = (1/8)[g\beta(T_w-T_\infty)\alpha^2x^3]^{-1/2}$$

$$Gr_x = g\beta(T_w-T_\infty)x^3/v^2,\quad Pr = v/\alpha. \tag{5}$$

If the velocity normal to the wall v_w is assumed to vary as $x^{1/4}$, then the mass transfer parameter F_w will be a constant and $F_w \gtrless 0$ according to whether it is a suction or an injection. The magnitude of the parameter λ determines the importance of the axial heat conduction effect. For natural convection flow in low-Prandtl-number fluids (such as a liquid metal), the local Grashof number Gr_x is large, but the Prandtl number Pr is small. However the product of Gr_x and Pr is assumed to be comparatively large such that $\lambda < 1$. Since λ is proportional to $x^{-3/2}$, the effect of axial heat conduction, which is large for small x, decreases as x increases. It may be remarked that for $\lambda = 0$ (in the absence of the axial heat conduction), the governing equations (3a) and (3b) reduce to those of classical natural convection flow which has been studied thoroughly by Minkowycz and Sparrow [2, 3], Cebeci [4], and Aziz and Na [5]. Also ξ is the transverse curvature parameter and $\xi = 0$ corresponds to the flat plate case.

The main focus of the presentation of results will be the local heat transfer which is defined as [2, 3]

$$Nu_x = -x(\partial T/\partial r)_w/(T_w-T_\infty) = -G'_w(Gr_x/4)^{1/4}.$$

RESULTS AND DISCUSSION

The governing equations (3a) and (3b) under boundary conditions (4) have been solved numerically using an implicit finite-difference scheme in combination with the quasilinearization technique [6]. In order to assess the accuracy of our method, we have compared our heat transfer results without axial heat conduction effect ($\lambda = 0$) with those of Minkowycz and Sparrow [2, 3], Cebeci [4], and Aziz and Na [5], and found them to be in very good agreement. The comparison is shown in Tables 1 and 2.

Table 1. Ratio of local Nusselt numbers, $(Nu_x)/(Nu_x)_{\lambda=0}$ for $\lambda = F_w = 0$, $Pr = 0.72$

ξ	Present results	Minkowycz and Sparrow [2]	Cebeci [4]	Aziz and Na [5]
0	1.0	1.000	1.000	1.000
0.503	1.2101	1.212	1.210	1.219
1.064	1.4219	1.428	1.422	1.445
2.093	1.7782	1.787	1.778	1.821
3.364	2.1769	2.170	2.177	2.232
4.000	2.3658	2.363	2.366	2.419
5.030	2.6605	2.674	2.660	2.770

Table 2. Values of local heat transfer $-G'_w$ for $\lambda = 0$, $Pr = 0.01$

ξ	F_w	Present results	Minkowycz and Sparrow [3]
0	−1.6	0.7183(−1)*	0.7182(−1)
	0	0.8057(−1)	0.8056(−1)
	1.6	0.8850(−1)	0.8849(−1)
1.0	−1.6	0.2361	0.2354
	0	0.2449	0.2441
	1.6	0.2532	0.2523
2.0	−1.6	0.3801	0.3779
	0	0.3885	0.3864
	1.6	0.3969	0.3947
5.0	−1.6	0.7788	0.7726
	0	0.7873	0.7810
	1.6	0.7965	0.7892

*0.7182(−1) = 0.7182 × 10^{-1}.

The effect of axial heat conduction parameter λ on the local heat transfer rate, $-G'_w$, for some representative values of Pr and ξ without mass transfer ($F_w = 0$) is shown in Fig. 1 and with mass transfer ($F_w \neq 0$) in Fig. 2. It is evident from these figures that the local heat transfer rate ($-G'_w$) increases as the axial heat conduction parameter (λ), Prandtl number (Pr), transverse curvature (ξ), and mass transfer (F_w) increase. The physical reason for such a behaviour is the reduction in the

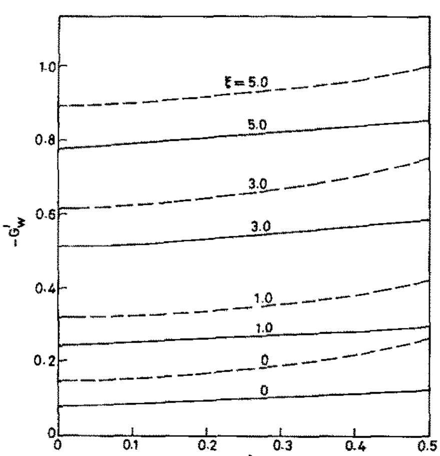

FIG. 1. Heat transfer $-G'_w$. —— $Pr = 0.01$; - - - - - $Pr = 0.05$.

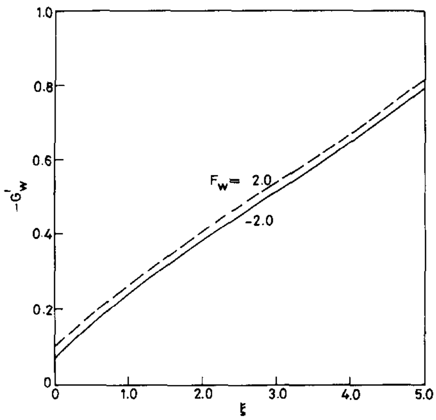

FIG. 2. Heat transfer $-G'_w$ for $\lambda = 0.2$ and $Pr = 0.01$. —— $F_w = -2.0$; ----- $F_w = 2.0$.

thickness of the thermal boundary layer due to increase in λ, Pr, ξ and F_w. For small Pr and ξ, the heat transfer $(-G'_w)$ is found to be strongly dependent on the heat conduction parameter, λ. For example, when $Pr = 0.05$ and $\xi = 0$, the local heat transfer rate for $\lambda = 0.5$ is found to be almost twice that of $\lambda = 0$. This implies that the effect of the axial heat conduction cannot be neglected for small Pr and ξ. The skin friction parameter F''_w is found to depend weakly on the axial heat conduction parameter, λ. Hence, it is not shown here. It can be seen from Fig. 2 that the heat transfer, $-G'_w$, is weakly dependent on the mass transfer, F_w, but the skin friction, F''_w, is appreciably affected by it. Since the skin friction results (F''_w) for $F_w \neq 0$ and $\lambda = 0$ are given in ref. [3], they are not presented here.

CONCLUSIONS

The heat transfer increases as the axial heat conduction parameter, Prandtl number, transverse curvature and mass transfer increase. The effect of the axial heat conduction on the heat transfer is found to be more pronounced for small curvature. The skin friction is found to be weakly dependent on the axial heat conduction parameter.

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